Analysis 1B — Week 11

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# Introduction

Here is the material to accompany Week 11 of the Analysis 1B course. Alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to [Christian Jones (caj50)](mailto:caj50@bath.ac.uk). To return to the homepage, click [here](http://caj50.github.io/tutoring.html).

# Lecture Recap

There’s no tutorial this week due to the May Day bank holiday. However, there’s still a bit of the course to summarise (including the fundamental theorem of calculus). So here, you’ll find some material that would have been covered if everything was running as normal.

## Some Integral Theorems

We begin with two theorems, which basically say that integrals behave as you’d expect them to.

Theorem 1.1 (Additivity of the Integral)

Let and let be bounded. For any , is integrable on if and only if is integrable on and In this case

We can see this theorem in action below in Figure 1.1.

Figure 1.1: The integral being additive means that to integrate a function f on a domain [a,b], we can just sum up the integrals of f on some smaller domains. This is especially useful for functions defined piecewise.

Figure 1.1: The integral being additive means that to integrate a function on a domain , we can just sum up the integrals of on some smaller domains. This is especially useful for functions defined piecewise.

Theorem 1.2 (Linearity of the Integral)

Let , and let be integrable. Then

1. is integrable, with
2. is integrable, with

Now is the time to bring some algebra into the mix. Since the zero function given by is integrable, this means that the set of integrable functions on is a vector subspace of the set of bounded functions on .

### Useful Results about and .

To prove the above two theorems, we need to know a bit about how the bounds of a sum of two bounded functions behave. The following results tell us exactly what we want!

Proposition 1.3

For non-empty, and bounded. Then:

* is bounded, with
* If :
* If :

### Some Other Useful Facts

Using everything we’ve learned so far, we can state a few more facts about integrals!

Proposition 1.4

If and are integrable, then:

1. (This is by definition!)
2. (This is also by definition!)
3. The function is integrable (recall )
4. is integrable, with

It’s worth mentioning here that point 4 above is actually an analogue of the triangle inequality for integrals!

## The Fundamental Theorem(s) of Calculus

We’ve finally made it to the biggest theorems of the course, and this ties in everything done in the last 11 weeks! Despite usually being called *the* fundamental theorem of calculus (FTC), it actually encompasses two statements, which is why we state them separately below:

Theorem 1.5 (Fundamental Theorem of Calculus I)

Let and be an open interval containing . Let be differentiable on , and let be such that

* ,[[1]](#footnote-36)
* is integrable on .

Then

Theorem 1.6 (Fundamental Theorem of Calculus II)

Let be integrable, and define via

Then

* is continuous on and
* If is continuous at , then is differentiable at with

So, why should we like these theorems so much? The first theorem makes finding integrals much easier, as derivatives are (generally) nicer to deal with than Riemann sums! The second theorem here gives you *existence* of a primitive for , which you can apply the first theorem to!

As a warning, if in Theorem 1.6 is **not** continuous at , may still be differentiable at ! See Problem Sheet 11, Tutorial Question 1 for more details.

## Integration Techniques

To finish the lecture recap, it would be handy to have some theorems which back up some standard integration techniques. Turns out that we do! The first of these (in a fashion) ‘undoes’ the differential product rule.

Theorem 1.7 (Integration By Parts)

Let be continuous. Suppose that are continuous on and differentiable on with and Then

And finally, the second of these can be seen as a way of ‘undoing’ the differential chain rule.

Theorem 1.8 (Integration By Substitution)

Let be a closed interval and let be continuous on . Also, let be an open interval, and let be continuously differentiable[[2]](#footnote-44). Then, for ,

# Hints

Whilst there’s not an official hand in this week, you may still find this section useful. So, here’s the hints for the extra questions on Problem Sheet 11.

1. Write , where From this point, the methods used in one of the tutorial questions from Problem Sheet 9 will come in handy!
2. Write the integral in the form of something you can apply the Fundamental Theorem of Calculus to. Make sure to verify the hypotheses!
3. See Q2) above.

1. This is the definition of being a *primitive* for . [↑](#footnote-ref-36)
2. This means that is differentiable on , and that the derivative is continuous on . For interest, the set of all continuously differentiable functions from a set is denoted by . [↑](#footnote-ref-44)